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Effects of preconditioning on the accuracy and efficiency of incompressible flows

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SUMMARY

An investigation of some form of preconditioning approach for the incompressible Navier–Stokes equations is presented. We have implemented preconditioning in conjunction with a high-resolution (characteristics-based) scheme for the advective terms, a non-linear multigrid algorithm and an explicit fourth-order, total variation diminishing (TVD) Runge–Kutta scheme. Computations have been carried out for flows through suddenly-expanded and expanded–contracted geometries, for a broad range of Reynolds numbers, featuring flow separation as well as instabilities. We present comparisons of the preconditioned and non-preconditioned solutions against experimental and previous computational results and show that for the cases exhibiting instabilities, preconditioning has a positive effect on the convergence, but the accuracy is adversely affected. Further investigations of other forms of preconditioning need to be performed in order to shed light on the above issues. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: preconditioning; incompressible flows; instabilities; high-resolution methods

1. INTRODUCTION

Preconditioning techniques in CFD aim to overcome stiffness in the solution of the equations [1, 2]. There are two main streams of research. Firstly, the development of preconditioning for low speed and incompressible flows. The artificial compressibility method of Chorin [3] can also be viewed as a preconditioning technique. Secondly, methods that aim to alleviate discrete stiffness in the Euler and Navier–Stokes equations, including clustering high frequency eigenvalues away from the origin, thus, providing rapid damping by a multi-stage scheme, directional coarsening multigrid and alternating direction implicit preconditioners. Preconditioning methods for the compressible equations have been investigated by several researchers; see Reference [2] for a review on this topic. They present generalizations of

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the incompressible artificial compressibility formulation to compressible equations. Turkel's approach modifies the transient behaviour of the Navier–Stokes equation in such a way that the stiffness is removed from the eigenvalues. Lee's and van Leer's [4] preconditioner uses a minimum range in the characteristic speeds and a minimum variation from the associated eigenvectors. Lynn [5] further developed the idea of Reference [4] and found that at stagnation points the preconditioner produced instabilities which could not be fixed.

Here, we investigate the effect of preconditioning on the accuracy and efficiency of incompressible flows. In general, we aim to investigate the circumstances in which preconditioning should be used. We have implemented Turkel's preconditioner [1] in conjunction with the artificial compressibility of the Navier–Stokes equations [3], a characteristics-based (high-resolution) scheme for the discretization of the advective terms [6, 7], an explicit, TVD fourth-order Runge–Kutta scheme [8] and a multigrid algorithm [9].

Our work concentrates on separated internal flows featuring instabilities that manifest as symmetry-breaking bifurcations (see Reference [10] and references therein, for example). The importance of understanding non-linear bifurcation phenomena in fluid mechanics is motivated by the quest to obtain a deeper understanding of hydrodynamic stability and laminar-to-turbulent transition. It is also equally important to understand the behaviour of numerical methods that are used to simulate such phenomena. Therefore, we have performed several numerical experiments for unstable, separated channel flows with and without the use of preconditioning. We show that preconditioning has a positive effect on the convergence (towards steady state solution) at low Reynolds numbers as long as the flow remains stable. However, the numerical studies reveal that preconditioning has an adverse effect on the accuracy of bifurcating unstable flows (still at low Reynolds numbers). This is shown by comparing the computations based on preconditioning against experimental results and computations without preconditioning.

2. NUMERICAL FRAMEWORK

We have employed the incompressible Navier–Stokes equations (for steady state flow problems) in their pseudo-compressible formulation

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j + p\delta_{ij}) = \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j^2} \tag{1}$$

$$\frac{1}{\beta}\frac{\partial p}{\partial \tau} + \frac{\partial u_j}{\partial x_j} = 0 \tag{2}$$

where u_i are the velocity components (the indices $i, j = 1, 2$ refer to the space co-ordinates x, y for a two-dimensional problem), p is the pressure and Re is the Reynolds number; all variables are properly normalized. For steady flows we obtain $\tau \equiv t$. The system is integrated in a pseudo-time τ to a steady state, assuming an artificial speed of sound $\sqrt{\beta}$, where β is the artificial compressibility parameter.

We have employed a high-resolution, Godunov-type method known as the characteristics-based scheme [6] for the discretization of the advective terms. The scheme is a flux averaging procedure according to which, the flow variables, and subsequently the fluxes, are calculated at the cell faces by a Godunov-type approach. In our model the default time integration with respect to τ employs a fourth-order TVD Runge–Kutta scheme [8] (selected, primarily, for the optimum performance on non-uniform grids) while a non-linear multigrid method [9] is used to accelerate the convergence towards the steady state.

The reconstructed characteristics-based variables [6, 7] are used in the calculation of the intercell fluxes. For a curvilinear co-ordinates system (ξ, η), the reconstructed variables associated with the advective flux in the ξ direction are given by

$$\tilde{\mathbf{U}} = \begin{pmatrix} \tilde{p} \\ \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} (\lambda_1 k_2 - \lambda_2 k_1)/2s \\ R\tilde{x} + \tilde{y}(u_0\tilde{y} - v_0\tilde{x}) \\ R\tilde{y} - \tilde{x}(u_0\tilde{y} - v_0\tilde{x}) \end{pmatrix} \tag{3}$$

where $s = \sqrt{{\lambda_0}^2 + \beta}$ $(\lambda_0 = u\tilde{x} + v\tilde{y})$; $\tilde{k} = \xi_k / \sqrt{\xi_x^2 + \xi_y^2}$ $(k = x, y)$ and

$$R = \frac{1}{2s}[p_1 - p_2 + \tilde{x}(\lambda_1 u_1 - \lambda_2 u_2) + \tilde{y}(\lambda_1 v_1 - \lambda_2 v_2)] \tag{4}$$

$$k_1 = p_1 + \lambda_1(u_1\tilde{x} + v_1\tilde{y}), \quad k_2 = p_2 + \lambda_2(u_2\tilde{x} + v_2\tilde{y}) \tag{5}$$

The variables p_l, u_l, v_l $(l = 0, 1, 2)$ are the primitive variables on the characteristics, which are calculated by a third-order Godunov-type interpolation [7]. The reader is referred to References [6, 7, 9] for further information regarding the numerical scheme and the multigrid approach.

Turkel's preconditioning approach‡ replaces the momentum equation with

$$\frac{(\alpha + 1)}{\beta} u_i \frac{\partial p}{\partial \tau} + \frac{\partial u_i}{\partial \tau} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j^2} \tag{6}$$

where α is yet another parameter controlling the attenuation of the flow divergence towards zero. Systems (2) and (6) are hyperbolic since their eigenvalues are real. For example, the eigenvalues associated with the momentum flux in the x-direction (in a Cartesian system) are given by

$$\lambda_0 = u, \quad \lambda_{1,2} = \frac{(1-\alpha)u \pm \sqrt{(1-\alpha)^2 u^2 + 4\beta}}{2} \tag{7}$$

The choice of β needs to be optimized to minimize the largest possible ratio of wave speeds. Turkel proposed the calculation of β as

$$\beta^2 = \begin{cases} \max[(2-\alpha)(u^2 + v^2), \varepsilon], & \alpha < 1 \\ K \max[\alpha(u^2 + v^2), \varepsilon], & \alpha \geqslant 1 \end{cases} \tag{8}$$

where β is a function of the fluid speed $(u^2 + v^2)$. The value of ε should be a fraction of $(u^2 + v^2)_{\max}$ and the value of K should be chosen slightly larger than one. In the present

‡Note that in Reference [1] the derivation was presented for the inviscid incompressible equations, but it can also be formally applied to the system of the Navier–Stokes equations.

investigation, we have carried out computations for a broad range of values for the parameters α and β (see discussion below). Note that the dissipation properties of the characteristics-based scheme will also alter as a result of the preconditioned eigenvalues.

3. RESULTS

We have carried out various numerical experiments for flows through suddenly expanded–contracted (SEC) and suddenly-expanded (SE) geometries at low Reynolds numbers. At certain Reynolds numbers these flows feature instabilities manifested in the form of a symmetry-breaking bifurcation. The geometries we have considered are a SEC channel for which experimental results (Figure 1) are available [11] and the (classical) SE channel problem for which there are published computational and experimental results [10, 12, 13].

Different computational grid sizes were employed and it was found that 30 000 and 25 000 grid points were sufficient to obtain grid independent solution for the expanded–contracted and SE channels, respectively. Computations have been carried out for a broad range of low Reynolds numbers spanning from 1 to 250, based on the maximum inlet velocity and upstream channel height; a parabolic inlet velocity profile was used in all cases. Both flow geometries lead to symmetric flow separation at lower Reynolds numbers and present a symmetry-breaking bifurcation as the Reynolds number increases. The expanded–contracted channel returns to a symmetric flow as the Reynolds number further increases [11].

For both flow geometries, the best convergence results were obtained for α values between 0 and 1. When $\alpha = -1$ the original artificial compressibility formulation is obtained, while for $\alpha = 1$ the eigenvalues λ_1 and λ_2 are only functions of β. In some cases, this has dissipative effects on the solution. It was found that preconditioning does not have any significant effect on the convergence at Reynolds numbers $Re < 10$, but it does have a positive effect at higher Reynolds numbers. In this case the total number of multigrid cycles can be reduced by 30%. The calculation of β as proposed by Turkel (Equation 8) was found not to work effectively when the Reynolds number was reduced below 20. A fixed β value was found to provide better convergence results (both for preconditioned and non-preconditioned solutions) and its precise values can significantly affect the convergence. For example, for a Reynolds

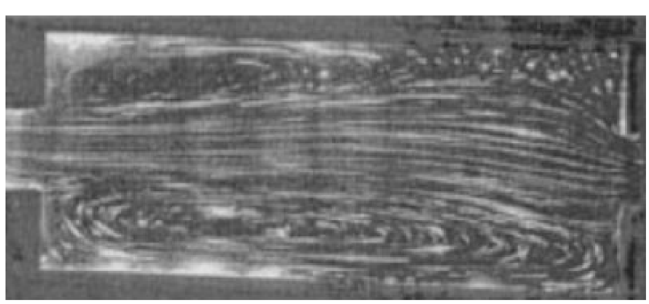

Figure 1. Experimental results at $Re = 116$ [11] for the suddenly contracted-expanded geometry. Reproduced with permission from Cambridge University Press.

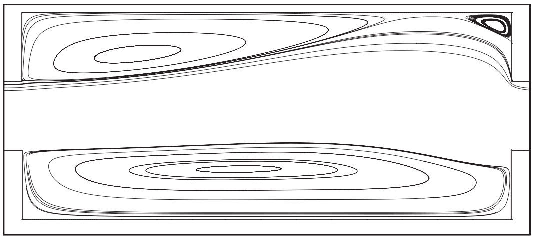

Figure 2. Non-preconditioned solution at $Re = 116$.

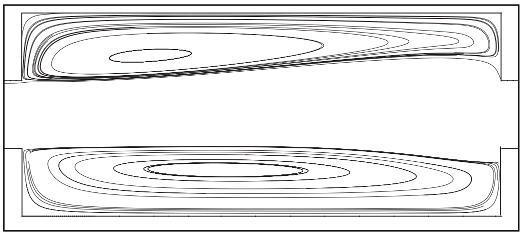

Figure 3. Preconditioned solution at $Re = 116$.

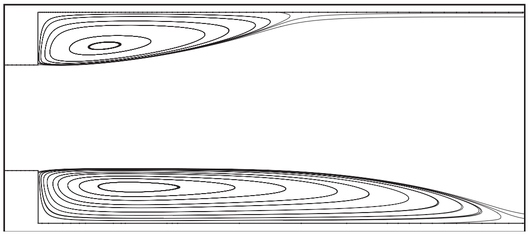

Figure 4. Non-preconditioned solution at $Re = 250$.

number of 10 and $\beta = 1$, the number of multigrid cycles needed to obtain a converged solution was approximately 1780. For $\beta = 0.8$ a converged solution was reached after 800 multigrid cycles.

However, the most important effects of preconditioning were found to be on the accuracy of the flow solution, especially in the range of Reynolds numbers where instability occurs. For lower Reynolds numbers, where the flow is symmetrically separated the accuracy of the solution was found not to be altered by the use of preconditioner for the entire range of α and β values used here.

Figures 2–5 as well as Tables I and II summarize the results of the preconditioned and non-preconditioned solutions. Table II provides comparisons between the present results and

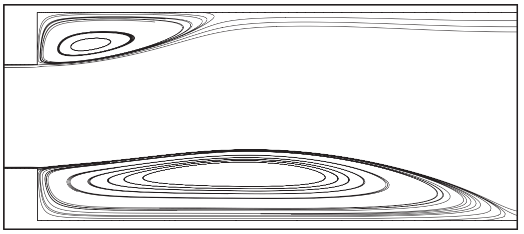

Figure 5. Preconditioned solution at $Re = 250$.

Table I. Results for preconditioned and non-preconditioned solutions for the two channel geometries (see text for details).

Case	Reynolds number	Preconditioned solution	Non-preconditioned solution
SEC	1–20	Symmetric	Symmetric
SEC	60	Symmetric (diffusive)	Symmetric
SEC	116	Asymmetric (diffusive)	Asymmetric
SEC	200	Asymmetric	Symmetric
SE	0.1–1	Symmetric	Symmetric
SE	100	Symmetric (diffusive)	Symmetric
SE	250	Asymmetric (diffusive)	Asymmetric

Table II. Comparison of solutions with previously published results [10–12] for the two geometries, for symmetric (S) and asymmetric (A) cases.

Case	Re	Published results from [10–12]	Preconditioned solution	Non-preconditioned solution
SEC	116	$\Delta x = 0.019$ m (A)	$\Delta x = 0$ (A)	$\Delta x = 0.018$ m (A)
SEC	200	Symmetric	Asymmetric	Symmetric
SE	100	Bubble size = 0.016 m (S)	Bubble size = 0.019 m (S)	Bubble size = 0.0159 m (S)
SE	250	$\Delta x = 0.02$ m (A)	$\Delta x = 0.027$ m (A)	$\Delta x = 0.02$ m (A)

previous experimental and computational studies. All the results refer to grid independent solutions. For the expanded–contracted channel, the experimental flow visualization clearly shows the occurrence of instability in the form of asymmetric separation. This is correctly predicted by the non-preconditioned solution, but not the second separation bubble on the upper right corner of the geometry. Further, for the same geometry at $Re = 200$ the flow again becomes symmetric but the preconditioned solution still remains asymmetric (Table I) with different sized bubbles on the lower and upper walls. At $Re = 116$, the computed distance Δx between the re-attachment points of the upper and lower bubbles without using preconditioner agrees well (Table II) with the experimental results of Reference [11], where in the preconditioned solution the bubble does not re-attach before it reaches the wall of the contraction part of the channel (i.e. $\Delta x = 0$ in Table II).

Similarly, the preconditioner has adverse effects on the accuracy of the suddenly-expanded channel flow. For example, at $Re = 250$ where the flow exhibits an instability (Figures 4 and 5) the preconditioned solution is qualitatively correct, but not with respect to the size of separation. Table II compares the present preconditioned and non-preconditioned solutions with the results of References [10, 12] for symmetric (stable) flow at $Re = 100$ and asymmetric (unstable) flow at $Re = 250$.

Parallel to this investigation, we have conducted several numerical experiments [13] using different Godunov-type methods without the use of a preconditioner. These investigations showed that more dissipative advective schemes generally lead to a stable flow, especially when the solution is under-resolved. Even though a rigorous analysis of the dissipation effects of non-linear approximations such as high-resolution Godunov-type schemes in combination with preconditioning appears very difficult, the similarity in the behaviour of certain Godunov-type schemes with the preconditioned results obtained here seems to indicate that preconditioning has an added dissipation effect on the solution, when the flow exhibits symmetry-breaking bifurcation. Furthermore, this kind of 'non-physical' behaviour exhibited by the preconditioned method at certain Reynolds numbers has also some similarities with the volatile numerical behaviour of some time integration methods, which perform differently depending on the solution problem [14].

4. CONCLUSIONS

A numerical study showing the effects of preconditioning on flows through SEC and SE channels was presented. Laminar flow calculations were performed with and without preconditioning in order to assess its effects on the accuracy and efficiency of computations. At higher Reynolds number flows the use of preconditioning reduced the number of multigrid cycles, but adversely affected the solution results. For Reynolds numbers in the range of instability, the use of preconditioning led to either an incorrect stable solution or to an under-estimation of the size of the separation bubble. At lower Reynolds number flows the present form of preconditioning neither altered the accuracy of the solution nor had a significant effect on the convergence. Finally, computational studies on the accuracy of other forms of preconditioning on unstable flows, e.g. by scaling the pseudo-acoustic speed according to the diffusive time scale [15], will be presented in a future paper.

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